

Combinatorics of generalized Bethe equations

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Abstract

A generalization of the Bethe ansatz equations is studied, where a scalar two-particle S-matrix has several zeroes and poles in the complex plane, as opposed to the ordinary single pole/zero case. For the repulsive case (no complex roots), the main result is the enumeration of all distinct solutions to the Bethe equations in terms of the Fuss-Catalan numbers. Two new combinatorial interpretations of the Fuss-Catalan and related numbers are obtained. On the one hand, they count regular orbits of the permutation group in certain factor modules over \mathbb{Z}^M , and on the other hand, they count integer points in certain M -dimensional polytopes.

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Introduction

The modern techniques of quantum integrability rely on what is known today as the *Bethe ansatz*, a powerful tool, first introduced by H. Bethe [1] and refined by many authors throughout the years, see the summarising monographs [2, 3, 4]. The idea behind the Bethe ansatz is to reduce, in one way or another, the spectral problem for the Hamiltonian in the M -particle sector to the one of solving a set of M simultaneous equations (algebraic or transcendental, depending on the model) for particles' *rapidities* v_n .

In the case when the particles carry no internal degrees of freedom (are spinless), those equations, referred to as *Bethe equations*, can be cast in the form

$$F(v_n) \prod_{\substack{m=1 \\ m \neq n}}^M S(v_n - v_m) = 1, \quad n = 1, \dots, M, \quad (0.1)$$

and interpreted as the periodicity conditions for the wave function of the system of M one-dimensional particles put in a periodic box, in the assumption that all multi-particle collisions can be reduced to a sequence of two-particle collisions (*factorisation of S -matrix* hypothesis). The function F can be interpreted then as a one-particle travel phase factor, and S as a two-particle scattering phase factor.

Most of the solutions of the equations (0.1), referred to in the present paper as *weak* generalized Bethe equations (weak GBE), correspond to the discrete eigenstates of the integrable system. To eliminate the nonphysical solutions one needs, as explained in section 1, to supplement (0.1) with additional conditions. Such corrected GBE will be called *strong* GBE.

Although F can take a wide range of forms depending on the chosen integrable model, the expression for S is much less variable, being restricted by the quantum group governing the integrability of the model. In the most of the presently known examples, the S -factor has only a single pole and a single zero in the complex plane (modulo periodicity for trigonometric or elliptic functions). The only exception known so far is the Izergin-Korepin model [5, 6].

In the present paper we start a systematic study of the Bethe equations, where the function S has K poles and K zeroes. Our primary interest is the case $K > 1$, referred to as the *generalized* Bethe equations (GBE), as opposed to the well-studied *ordinary* case $K = 1$. Our main motivation comes from the recent progress on AdS/CFT correspondence in string theory that produced a similar kind of Bethe equations [7, 8]. It has to be emphasized that in contrast to the “asymptotic Bethe ansatz” approach as in [7, 8], where the equations (0.1) hold in the infinite volume limit only, we assume here that they are satisfied exactly in the finite volume as well.

We shall concentrate on the simplest task: counting the number of solutions to the strong Bethe equations. Even in the ordinary case of a single pole/zero S -function, the combinatorics of Bethe ansatz related to the so-called “string hypothesis” and “completeness problem” has been found to be rewardingly rich and interesting [9, 10]. Mathematically, the task is to count solutions of a special system of algebraic equations. We hope that our results present interest not only for the specialists in integrable systems but also in combinatorics.

To avoid complications caused by complex solutions we focus on the *repulsive* case, when all roots v_n are real. In this case, the *strong* conditions are equivalent simply to discarding multiple roots of the Bethe equations. We consider two cases: the functions F and S being ratios of trigonometric or ordinary polynomials of degree N for F and degree K for S . The number of distinct solutions to the strong GBE is then denoted $C_M^{(K,N)}$.

To count the solutions of GBE we use the fact that they correspond to the minima of certain convex function (Yang-Yang function). In the trigonometric case, we establish a one-to-one correspondence between the set of distinct solutions to the GBE and the set of regular orbits of the permutation group \mathfrak{S}_M in certain factor modules over \mathbb{Z}^M . In the rational case, we bijectively label the solutions with the integer sequences $I_1 < \dots < I_M$ satisfying certain linear inequalities. For a more detailed description see respectively theorems 1 and 2 below.

For the numbers $C_M^{(K,N)}$ that enumerate both combinatorial objects and therefore the solutions of strong GBE as well, we then obtain an explicit formula

$$C_M^{(K,N)} = \frac{N}{M} \binom{KM + N - 1}{M - 1} = \frac{N}{KM + N} \binom{KM + N}{M} \quad (0.2)$$

that constitutes the main result of our paper.

The respective generating function turns out to be the N^{th} power of the generating function for the Fuss-Catalan numbers [11, 12] corresponding to $C_M^{(K,1)}$ in our notation:

$$C^{(K,N)}(t) = \sum_{M=0}^{\infty} C_M^{(K,N)} t^M = \left(C^{(K,1)}(t) \right)^N. \quad (0.3)$$

The particular case $C_M^{(2,1)}$ corresponds to the famous Catalan numbers [13] that are known to have over 120 different combinatorial interpretations [14].

Two theorems formulated below provide two new combinatorial interpretations of the numbers $C_M^{(K,N)}$, including Fuss-Catalan numbers $C_M^{(K,1)}$ and Catalan numbers $C_M^{(2,1)}$ as particular cases, obtained as a result of our study of GBE.

Theorem 1. *Let N , M and K be some positive integers. Two M -tuples of integers (I_1, \dots, I_M) are considered equivalent if one can be mapped into another by a finite composition of basic transformations γ_n ($n = 1, \dots, M$)*

$$\gamma_n : I_n \mapsto I_n + N + (M - 1)K \quad \text{and} \quad \gamma_n : I_m \mapsto I_m - K \quad m \neq n, \quad (0.4)$$

An equivalence class is called regular if it contains only M -tuples of pairwise distinct integers (I_1, \dots, I_M) .

Then the number of all regular equivalence classes is given by $M! C_M^{(K,N)}$.

Theorem 2. *Let N , M and K be some positive integers, then the number of different choices of strictly increasing integer sequences $I_1 < \dots < I_M$ satisfying to $2M$ constraints*

$$\frac{r(r-1)}{2} K \leq I_1 + I_2 + \dots + I_r, \quad (0.5a)$$

$$I_{M-r+1} + \dots + I_{M-1} + I_M \leq rMK - \frac{r(r+1)}{2} K + rN - 1, \quad (0.5b)$$

for $r = 1, \dots, M$ equals $C_M^{(K,N)}$.

The paper is organized as follows. In Section 1, we give a precise description of the class of functions F and S that we consider. We then define the strong generalized Bethe equations (GBE) and prove certain elementary properties of the solutions to the weak and strong GBE associated with our choice of F and S . In Section 2, we reformulate GBE as the minimality condition for a convex Yang-Yang function and establish a one-to-one correspondence between the set of distinct solutions to the strong GBE and the sets of integers described in theorems 1 and 2. Then, we distinguish between the trigonometric and rational cases. In Section 3, we apply the theory of commutative rings to count the distinct solutions to the strong GBE in the trigonometric case and prove theorem 1. Finally, in Section 4, we analyze the limit when the period of trigonometric functions goes to infinity and show that the number of solutions in the rational case is the same as in the trigonometric case. Theorem 2 is then obtained as a corollary of the established bijection.

The important question as to what kind of integrable models and what quantum groups might correspond to our generalized Bethe equations is left unresolved. We plan to address it in our subsequent publications.

1 The general setting

Throughout the paper, we consider two cases of interest. We assume that F and S are either realized as ratios of ordinary polynomials or as ratios of trigonometric polynomials. We will refer to the first case as the *rational* case and to the second one as the *trigonometric* case.

Let $\psi(u) = u$ in the rational case and $\psi(u) = \sin(u)$ in the trigonometric one. The functions F and S are characterized by natural numbers N, K and complex parameters $F_\infty, \xi_n, \eta_j \in \mathbb{C}$ and are defined as

$$F(u) = F_\infty \prod_{n=1}^N \frac{\psi(u + \xi_n)}{\psi(u + \bar{\xi}_n)}, \quad |F_\infty| = 1, \quad \text{Im}(\xi_n) > 0, \quad n \in \{1, \dots, N\}, \quad (1.1a)$$

$$S(u) = \prod_{j=1}^K \frac{\psi(u + \eta_j)}{\psi(u + \bar{\eta}_j)}, \quad \text{Im}(\eta_j) > 0, \quad j \in \{1, \dots, K\}. \quad (1.1b)$$

Here \bar{z} stands for the complex conjugate of z . The parameters η_j are chosen in such a way that the sets of zeroes and poles of S are both symmetric with respect to the imaginary axis:

$$\{\eta\} = \{-\bar{\eta}\}. \quad (1.2)$$

To avoid degeneracy of solutions to GBE, we demand that $F_\infty \neq 1$, or, equivalently, $F_\infty = e^{-2i\pi\varphi_\infty}$, with $-1 < \varphi_\infty < 0$. Later, in some cases we impose more restrictive conditions on φ_∞ .

Finally, in the trigonometric case, we also assume that the parameters ξ_n and η_j are located in the strip $-\pi/2 < \text{Re}(z) \leq \pi/2$.

The above restrictions on F and S are chosen in such a way that the phase factors, both trigonometric and rational, have the following natural, for a physicist, properties that are easy to derive from (1.1) and (1.2).

1. Unitarity.

$$\overline{F(u)} = F(\bar{u})^{-1}, \quad \overline{S(u)} = S(\bar{u})^{-1}, \quad u \in \mathbb{C}. \quad (1.3)$$

2. Parity. S respects the space-, or P -parity:

$$S(-u) = S(u)^{-1}, \quad (1.4a)$$

and the time-, or T -parity:

$$\overline{S(u)} = S(-\bar{u}). \quad (1.4b)$$

3. Repulsion condition:

$$|F(u)| > 1, \quad |S(u)| > 1, \quad \text{Im } u > 0, \quad (1.5a)$$

$$|F(u)| < 1, \quad |S(u)| < 1, \quad \text{Im } u < 0. \quad (1.5b)$$

Remark. The unitarity condition implies that $|F(u)| = 1 = |S(u)|$ for $u \in \mathbb{R}$. Also it is easy to see that the two parity conditions (1.4a) and (1.4b) are equivalent as long as one assumes unitarity.

It is a well known fact in the literature on Bethe ansatz that the Bethe equations (0.1), taken literally as they are written, have too many “false” solutions that do not correspond to any physical states. A common trick to eliminate them from consideration is to replace (0.1) with a different, more restrictive problem for v_m in terms of the Baxter polynomial:

$$Q(u|\mathbf{v}) \equiv \prod_{m=1}^M \psi(u - v_m), \quad \mathbf{v} = (v_1, \dots, v_M). \quad (1.6)$$

The problem for $Q(u|\mathbf{v})$ described below generalizes Baxter’s TQ equation known in the ordinary $K = 1$ case [2]. An important difference is that for $K > 1$ the equation for Q is not linear.

Let $\mathbf{z} = (z_1, \dots, z_M) \in \mathbb{C}^M$ and $\mathcal{Q}_{\{\eta\},\{\xi\}}(u|\mathbf{z})$ be defined as

$$\mathcal{Q}_{\{\eta\},\{\xi\}}(u|\mathbf{z}) = \alpha(u|\{\xi\}) \prod_{k=1}^K Q(u + \eta_k|\mathbf{z}) + \delta(u|\{\xi\}) \prod_{k=1}^K Q(u + \bar{\eta}_k|\mathbf{z}). \quad (1.7)$$

with

$$\alpha(u|\{\xi\}) = F_\infty \prod_{n=1}^N \psi(u + \xi_n), \quad \delta(u|\{\xi\}) = (-1)^{K+1} \prod_{n=1}^N \psi(u + \bar{\xi}_n). \quad (1.8)$$

We define the *strong* GBE to be the set of M conditions on the coordinates of the vector $\mathbf{v} = (v_1, \dots, v_m)$ necessary and sufficient that the function

$$u \mapsto \frac{\mathcal{Q}_{\{\eta\},\{\xi\}}(u|\mathbf{v})}{Q(u|\mathbf{v})} \quad (1.9)$$

is entire.

One can see that the above definition implies that $\mathcal{Q}_{\{\eta\},\{\xi\}}(u|\mathbf{v})$ vanishes at $u = v_j$ and hence the roots v_j always satisfy the weak GBE (0.1). However, the definition of the strong GBE is slightly more restrictive as it provides additional equations in the case of multiple roots (*ie* $v_j = v_k$ for some $j \neq k$).

1.1 General properties of solutions

In this subsection, we discuss general properties of the solutions to the weak and strong GBE with F and S given by (1.1). We first show that any solution to the weak, and hence to the strong GBE as well, has necessarily real roots v_j . Then, we show that for the strong GBE there are no “double” roots (ie $v_j \neq v_k$ for $j \neq k$). For the both properties the repulsion condition (1.5) is crucial. The proofs follow closely the standard ones [3, 4] for the case $K = 1$. Note that, until section 3, we ignore the π -periodicity in the trigonometric case and assume that $v_j \in \mathbb{R}$. If in a pair of solutions \mathbf{v} and \mathbf{v}' we have $v'_n = v_n + \pi$ the solutions are treated as different ones.

Proposition 1. *Let F and S satisfy the repulsion conditions (1.5). Then all of the coordinates v_n of a solution $\mathbf{v} = (v_1, \dots, v_M)$ to the weak GBE (0.1) are real.*

Proof. The proof goes as in [3]. One orders the roots v_n with respect to their increasing imaginary part:

$$\text{Im } v_1 \leq \text{Im } v_2 \leq \dots \leq \text{Im } v_M. \quad (1.10)$$

Then $\text{Im}(v_M - v_m) \geq 0 \ \forall m$ and therefore $|S(v_M - v_m)| \geq 1$ due to the repulsion condition (1.5). The GBE (0.1) then imply $|F(v_M)| \leq 1$ that is $\text{Im } v_M \leq 0$, as F satisfies the repulsion condition. Similarly, setting $n = 1$, we obtain that $\text{Im}(v_1 - v_m) \leq 0 \ \forall m$ leading to $|S(v_1 - v_m)| \leq 1$ and therefore $|F(v_1)| \geq 1$, that is to say $\text{Im } v_1 \geq 0$.

The two inequalities $\text{Im } v_M \leq 0$ and $\text{Im } v_1 \geq 0$ along with (1.10) lead to $\text{Im } v_n = 0$ for all n . ■

Proposition 2. [The exclusion principle] *Let F, S be as in (1.1), and \mathbf{v} be a solution to the strong GBE. Then the coordinates of \mathbf{v} are pairwise distinct, ie $v_j \neq v_k$ for $j \neq k$.*

Proof. Again, we follow closely a proof from [3] for the case $K = 1$. Assume that there exists a solution $\mathbf{v} = (v_1, \dots, v_M)$ to the strong GBE with a double root $v_k = v_j$ for some $j \neq k$. The condition for the expression (1.9) to be entire implies that $u = v_k$ is a double root of $\mathcal{Q}_{\{\eta\}, \{\xi\}}(u | \mathbf{v})$ (1.7) and thus

$$\left. \frac{d}{du} \cdot \mathcal{Q}_{\{\eta\}, \{\xi\}}(u | \mathbf{v}) \right|_{u=v_k} = 0. \quad (1.11)$$

By proposition 1, all the roots v_k are real. Therefore,

$$\alpha(v_k | \{\xi\}) \prod_{\ell=1}^K \prod_{m=1}^M \psi(v_k - v_m + \eta_\ell) \neq 0,$$

and one can apply the weak GBE (0.1) to recast the derivative of $\mathcal{Q}_{\{\eta\}, \{\xi\}}$ as

$$\begin{aligned} \left. \frac{d}{du} \cdot \mathcal{Q}_{\{\eta\}, \{\xi\}}(u | \mathbf{v}) \right|_{u=v_k} &= -\alpha(v_k | \{\xi\}) \prod_{\ell=1}^K \prod_{m=1}^M \psi(v_k - v_m + \eta_\ell) \\ &\times \left\{ \sum_{n=1}^N \frac{\psi(2i \text{Im } \xi_n)}{|\psi(v_k + \xi_n)|^2} + \sum_{\ell=1}^K \sum_{m=1}^M \frac{\psi(2i \text{Im } \eta_\ell)}{|\psi(v_k - v_m + \eta_\ell)|^2} \right\}. \end{aligned} \quad (1.12)$$

The factor appearing in the second line is purely imaginary with a strictly positive imaginary part and the pre-factor does not vanish. This contradicts the assumption that $\mathcal{Q}'_{\{\eta\},\{\xi\}}(v_k | \mathbf{v}) = 0$. \blacksquare

We have thus proven that in the repulsive case the strong GBE are equivalent to the weak GBE (0.1) supplemented with the condition $v_j \neq v_k$ for $j \neq k$. From now on, we can focus on the analysis of the weak GBE (0.1) only, simply discarding the solutions with multiple roots.

From the physical point of view, the solutions (v_1, \dots, v_M) differing by a permutation of roots v_j are indistinguishable, defining the same physical state. In the repulsive case, when all roots are real and distinct, it is sufficient then to consider only the *ordered* solutions that satisfy

$$v_1 < v_2 < \dots < v_M. \quad (1.13)$$

In this text, however, we do not assume condition (1.13), unless stated otherwise. The solutions satisfying (1.13) are always explicitly referred to as *ordered* solutions.

1.2 Logarithmic Bethe equations

The logarithmic Bethe equations [3, 4] provide a way of rewriting the Bethe equations in such a way that a given solution $\mathbf{v} = (v_1, \dots, v_M)$ becomes labelled by a set of M integers I_1, \dots, I_M . These equations (logBE) are obtained by taking the logarithm of (0.1):

$$\varphi(v_n) + \sum_{\substack{m=1 \\ m \neq n}}^M \theta(v_n - v_m) = 2\pi I_n, \quad n = 1 \dots M, \quad (1.14)$$

with φ and θ related to the logarithms of F and S

$$\varphi(u) \equiv i \ln F(u) \quad \text{and} \quad \theta(u) \equiv i \ln S(u). \quad (1.15)$$

We fix the branch of the logarithm of $S(u)$ by demanding that $\theta(0) = \pi K$, which is possible due to the symmetry (1.2). As for the branch of $\ln F(u)$, we just assume that it is fixed in some arbitrary way. We also stress that φ and θ are strictly increasing functions

$$\varphi'(u) = -i \sum_{n=1}^N \frac{\psi(2i \operatorname{Im} \xi_n)}{|\psi(u + \xi_n)|^2} > 0 \quad \text{and} \quad \theta'(u) = -i \sum_{\ell=1}^K \frac{\psi(2i \operatorname{Im} \eta_\ell)}{|\psi(u + \eta_\ell)|^2} > 0. \quad (1.16)$$

Moreover, due to (1.2), θ' is even.

To be able to use the above correspondence of the solutions to GBE and M -tuples of integers we need to describe the class of admissible integers I_n that can arise in the right-hand side of (1.14) and to establish a bijection between those M -tuples and the solutions to GBE.

The construction is different in the trigonometric and rational cases and is described in the next section. In the rest of the current section, we present the properties of the integers I_n that are shared by the trigonometric and rational cases.

Proposition 3. [Monotonicity principle]. *Under the repulsion conditions (1.5), a growing sequence of roots v_n of the logBE (1.14) corresponds to a growing sequence of integers I_n , and vice versa:*

$$v_k > v_j \iff I_k > I_j. \quad (1.17)$$

Proof. Suppose that $v_2 > v_1$. From the strict monotonicity (1.16) of $\varphi(u)$, resp. $\theta(u)$, it follows that $\varphi(v_2) > \varphi(v_1)$, resp. $\theta(v_2 - v_1) > \theta(v_1 - v_2)$ and $\theta(v_2 - v_m) > \theta(v_1 - v_m)$ for $m > 2$. Adding all the inequalities together we obtain $I_2 > I_1$.

Now assume that $I_2 > I_1$. If $v_1 = v_2$ then clearly $I_1 = I_2$, that contradicts the assumption. Hence $v_1 \neq v_2$. Also, the variant $v_2 < v_1$ is impossible as, due to the above paragraph, it would imply that $I_1 > I_2$. Therefore $v_2 > v_1$. ■

Proposition 4. *Let $\mathbf{I} \equiv (I_1, \dots, I_M)$ be M integers such that there exists a solution $\mathbf{v} = (v_1, \dots, v_M)$ to the logBE (1.14). Then \mathbf{v} is the only solution corresponding to the same \mathbf{I} .*

Proof. Note that the logBE (1.14) appear as the extremum condition $w_n \equiv \partial W / \partial v_n = 0$ for the potential $W(\mathbf{v})$ on \mathbb{R}^M :

$$W(\mathbf{v}) \equiv \sum_{n=1}^M \Phi(v_n) + \frac{1}{2} \sum_{m,n=1}^M \Theta(v_n - v_m) - 2\pi \sum_{n=1}^M I_n v_n + \pi(M-1)K \sum_{n=1}^M v_n, \quad (1.18)$$

where Φ and Θ are the integrals of φ and θ (1.15)

$$\Phi(u) \equiv \int_0^u \varphi(t) dt, \quad \Theta(u) \equiv \int_0^u \theta(t) dt. \quad (1.19)$$

The calculation of w_n uses the identity $\theta(u) + \theta(-u) = 2\theta(0) = 2\pi K$ that follows from θ' being an even function.

To prove the uniqueness of solutions to (1.14) for a given choice of \mathbf{I} , it suffices to show that the potential is strictly convex, and therefore, if it admits a minimum, the latter is unique. To prove the strict convexity of W it is enough to check that the Hessian of W

$$\frac{\partial^2 W}{\partial v_n \partial v_p} = \frac{\partial w_n}{\partial v_p} = \delta_{np} \left(\varphi'(v_n) + \sum_{m=1}^M \theta'(v_n - v_m) \right) - \theta'(v_n - v_p). \quad (1.20)$$

is a positively definite matrix. Evaluating the quadratic form of the Hessian (1.20) on an arbitrary nonzero vector $\mathbf{g} \in \mathbb{R}^M$ we obtain

$$\begin{aligned} \sum_{n,p=1}^M \frac{\partial^2 W}{\partial v_n \partial v_p} g_n g_p &= \sum_{n=1}^M g_n^2 \left(\varphi'(v_n) + \sum_{m=1}^M \theta'(v_n - v_m) \right) - \sum_{n,p=1}^M \theta'(v_n - v_p) g_n g_p \\ &= \sum_{n=1}^M \varphi'(v_n) g_n^2 + \sum_{n>m}^M \theta'(v_n - v_m) (g_n - g_m)^2 \geq \sum_{n=1}^M \varphi'(v_n) g_n^2 > 0. \end{aligned} \quad (1.21)$$

Here we used that $\varphi' > 0$ and $\theta' > 0$ on \mathbb{R} , and that θ' is even. ■

In the $K = 1$ case the potential W is of course the well-known Yang-Yang function [3].

2 Solvability of the generalized Bethe equations

In this section we derive the necessary and sufficient conditions on the integers I_j for the corresponding GBE to have a (unique, by proposition 4) solution. The problem of counting of the solutions to GBE will be thus reduced to the enumeration of the allowed sets of integers. The description of those sets is different in the trigonometric and rational cases.

2.1 Trigonometric case

It is useful to introduce an auxiliary function:

$$\chi(u, c) = i \ln \frac{\sin(u + ic)}{\sin(u - ic)} = \int_{-\frac{\pi}{2}}^u \frac{\sinh(2c)}{\sin(v + ic) \sin(v - ic)} dv, \quad (2.1)$$

fixing the branch of logarithm by the condition $\chi(-\pi/2) = 0$. Note that $u \mapsto \chi(u, c)$ is π quasi-periodic:

$$\chi(u + \pi k, c) = 2\pi k + \chi(u, c), \quad k \in \mathbb{Z} \quad \text{and such that} \quad \chi(-u, c) + \chi(u, c) = 2\pi. \quad (2.2)$$

Moreover, $\chi(u, c)$ grows monotonously on the real axis and its integral has the asymptotic behavior

$$\int_0^u \chi(t, c) dt = u^2 + O(u), \quad u \rightarrow \pm\infty. \quad (2.3)$$

The functions $\chi(u, c)$ allow us to provide a more explicit representation for φ and θ :

$$\varphi(u) = 2\pi\varphi_\infty + \sum_{n=1}^N \chi(u + \operatorname{Re}(\xi_n), \operatorname{Im}(\xi_n)), \quad \text{where} \quad F_\infty = e^{-2\pi i \varphi_\infty}, \quad (2.4a)$$

$$\theta(u) = \sum_{j=1}^K \chi(u + \operatorname{Re}(\eta_j), \operatorname{Im}(\eta_j)). \quad (2.4b)$$

One can check using (2.2) and (1.2) that indeed one has $\theta(0) = \pi K$.

We are now in position to prove the existence of solutions in the trigonometric case.

Proposition 5. *Let $\mathbf{I} = (I_1, \dots, I_M) \in \mathbb{Z}^M$ be any sequence of M integers. Then, in the trigonometric case where $\psi(u) = \sin(u)$, there exists a unique solution to the logBE.*

Proof. We have already seen in proposition 4 that, if a solution exists, then it is unique as it provides the minimum for a strictly convex potential W (1.18). To prove the existence of solutions, it is thus enough to show that W grows at infinity. Indeed, W is then bounded from below and hence has a local minimum.

It follows from (2.3) that Φ and Θ , the integrals of φ and θ , have the asymptotic behavior

$$\Phi(u) = \int_0^u \varphi(t) dt = Nu^2 + O(u), \quad \Theta(u) = \int_0^u \theta(t) dt = Ku^2 + O(u) \quad \text{for} \quad u \rightarrow \pm\infty. \quad (2.5)$$

Hence, it follows that $W(\mathbf{v})$ has the asymptotics

$$W(\mathbf{v}) = N \sum_{n=1}^M v_n^2 + \frac{K}{2} \sum_{m,n=1}^M (v_n - v_m)^2 + O(\max_j |v_j|). \quad (2.6)$$

The potential W thus grows in every direction at infinity, and hence W has a local minimum. The existence of solutions to (1.14) is thus proven. \blacksquare

So far we ignored completely the periodicity of trigonometric functions. However, the periodicity $F(u + \pi) = F(u)$, $S(u + \pi) = S(u)$ means that two different solutions $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^M$ of the logarithmic Bethe equations (1.14) are equivalent as solutions of the weak GBE (0.1) if $\mathbf{v} - \mathbf{v}' \in (\pi\mathbb{Z})^M$. To obtain a characterization of the set of distinct solutions one needs to express the equivalence of the vectors \mathbf{v} modulo the lattice $(\pi\mathbb{Z})^M$ in terms of the integers \mathbf{I} . The answer is given below.

Denote

$$a \equiv N + (M - 1)K, \quad b \equiv -K. \quad (2.7)$$

Proposition 6. *Let two solutions $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^M$ of the logBE (1.14) differ by the shift of a single root v_n by π :*

$$v'_m = v_m + \pi\delta_{mn}, \quad m = 1, \dots, M. \quad (2.8)$$

Then for the corresponding integer vectors we have

$$I'_m = I_m + a\delta_{mn} + b(1 - \delta_{mn}), \quad m = 1, \dots, M. \quad (2.9)$$

Proof. The statement follows from the logBE (1.14) and from the quasi-periodicity conditions

$$\varphi(u + \pi) = 2\pi N + \varphi(u), \quad \theta(u + \pi) = 2\pi K + \theta(u). \quad (2.10)$$

resulting from (2.2) and (2.4). \blacksquare

Corollary 1. *The set of distinct ordered in the sense (1.13) solutions of the trigonometric weak GBE (0.1) is in one-to-one correspondence with the equivalence classes $\mathcal{I} \in \mathbb{Z}^M / \gamma$ of integer vectors $\mathbf{I} \in \mathbb{Z}^M$ factorised over the lattice of the periods γ generated by the basis γ_n ($n = 1, \dots, M$):*

$$(\gamma_n)_m = a\delta_{mn} + b(1 - \delta_{mn}), \quad \gamma = \left\{ \sum_{n=1}^M c_n \gamma_n, \quad c_n \in \mathbb{Z} \right\}. \quad (2.11)$$

$$\mathbf{I} \approx \mathbf{I}' \iff \mathbf{I} - \mathbf{I}' \in \gamma. \quad (2.12)$$

Having thus characterized the distinct ordered solutions to the weak GBE in terms of the integer vectors \mathbf{I} we proceed to characterize the solutions to the strong GBE.

Definition 1. *An equivalence class $\mathcal{I} \in \mathbb{Z}^M / \gamma$ is called regular if all its representatives \mathbf{I} lie off diagonals of \mathbb{Z}^M , that is $I_m \neq I_n$ for $m \neq n$.*

Theorem 3. *The set of distinct ordered solutions to the strong GBE in the trigonometric case is in a one-to-one correspondence with the set of regular equivalence classes \mathcal{I} .*

Proof. The exclusion principle (proposition 2) demands to disregard the solutions \mathbf{v} of the logBE (1.14) having multiple roots. More precisely, we discard the whole equivalence class $\mathcal{V} \in \mathbb{R}^M/(\pi\mathbb{Z})^M$ if it contains a representative $\mathbf{v} \in \mathcal{V}$ such that $v_m = v_n$ for some $m \neq n$. Indeed, in such a case, by the monotonicity principle (proposition 3), for the corresponding integer vector \mathbf{I} we have $I_m = I_n$ for the same $m \neq n$. Eliminating the equivalence classes generated by such representatives \mathbf{I} we are left with the regular classes \mathcal{I} . Vice versa, by the same monotonicity principle, if \mathcal{I} is a regular class, all its representatives \mathbf{I} give rise to solutions \mathbf{v} with distinct coordinates. ■

2.2 The rational case

We now discuss the existence of solutions in the rational case. We will show that the condition of existence of solutions leads to a quite different description of the set of admissible integers \mathbf{I} leading to distinct solutions of the rational GBE.

We start by introducing the rational analog of the function $\chi(u, c)$:

$$\rho(u, c) = i \ln \left(\frac{u + ic}{u - ic} \right) = \int_{-\infty}^u \frac{2c}{v^2 + c^2} dv, \quad c > 0. \quad (2.13)$$

It satisfies $\rho(u, c) + \rho(-u, c) = 2\pi$ and has asymptotics

$$\rho(u, c) = \begin{cases} 0 - 2cu^{-1} + O(u^{-3}), & u \rightarrow -\infty \\ 2\pi - 2cu^{-1} + O(u^{-3}), & u \rightarrow +\infty \end{cases}. \quad (2.14)$$

The integral of ρ is explicitly computable and reads

$$\int_0^u \rho(t, c) dt = \pi u + 2u \arctan \frac{u}{c} - c \ln \left(1 + \frac{u^2}{c^2} \right) = \begin{cases} -2c \ln |u| + O(1), & u \rightarrow -\infty \\ 2\pi u - 2c \ln u + O(1), & u \rightarrow +\infty \end{cases}. \quad (2.15)$$

In the rational case, φ and θ are given by

$$\varphi(u) = 2\pi\varphi_\infty + \sum_{n=1}^N \rho(u + \operatorname{Re}(\xi_n), \operatorname{Im}(\xi_n)), \quad \theta(u) = \sum_{j=1}^K \rho(u + \operatorname{Re}(\eta_j), \operatorname{Im}(\eta_j)). \quad (2.16)$$

Again $\theta(0) = \pi K$ follows from (1.2). We also remind that $F_\infty = e^{-2i\pi\varphi_\infty}$.

Proposition 7. *Assume that the phase φ_∞ is such that*

$$-1 < (M+1)\varphi_\infty < 0. \quad (2.17)$$

then there exists a solution to the logarithmic GBE in the rational case if and only if for any subset $\mathcal{J} \subset \{1, \dots, M\}$ of cardinality $|\mathcal{J}| = r$ with $r = 1, \dots, M$, the M integers (I_1, \dots, I_M) satisfy the set of 2^M inequalities:

$$\frac{r(r-1)}{2} K \leq \sum_{j \in \mathcal{J}} I_j \leq rMK - \frac{r(r+1)}{2} K + rN - 1. \quad (2.18)$$

Corollary 2. (follows from the above result combined with propositions 1-4). *The distinct ordered solutions to the rational generalized Bethe equations are in one-to-one correspondence with monotonously increasing sequences of M integers $I_1 < \dots < I_M$ satisfying the set of 2^M constraints (2.18).*

Proof [Proposition 7]. Similarly to the proof of proposition 5, it is enough to prove the divergence of the potential W at infinity so as to ensure the existence of solutions. Reciprocally, if one is able to show that the potential is unbounded from below, then it cannot have a local minimum. Indeed due to the strict convexity of W , the local minimum would be global, contradicting the unboundedness of W .

We first assume that the set of 2^M conditions (2.18) holds and prove that W goes to $+\infty$ along any ray. This is enough to prove that W admits at least one minimum.

Next, we assume that at least one of the 2^M conditions (2.18) does not hold. Then we construct a ray along which the potential goes to $-\infty$. As discussed above, this implies that W cannot have any local minimum.

Suppose that the M integers I_1, \dots, I_M satisfy to the constraints (2.18). It follow from the asymptotics (2.15) of the antiderivative of $\rho(u, c)$, that $\Phi(u)$ and Θ have the asymptotics

$$\Phi(u) \equiv \int_0^u \varphi(t) dt = \begin{cases} 2\pi\varphi_\infty u - 2 \left(\sum_{j=1}^N \text{Im}(\xi_j) \right) \ln |u| + O(1), & u \rightarrow -\infty \\ 2\pi(\varphi_\infty + N)u - 2 \left(\sum_{j=1}^N \text{Im}(\xi_j) \right) \ln u + O(1), & u \rightarrow +\infty \end{cases},$$

$$\Theta(u) \equiv \int_0^u \theta(t) dt = \begin{cases} -2 \left(\sum_{j=1}^N \text{Im}(\eta_j) \right) \ln |u| + O(1), & u \rightarrow -\infty \\ 2\pi K u - 2 \left(\sum_{j=1}^N \text{Im}(\eta_j) \right) \ln u + O(1), & u \rightarrow +\infty \end{cases}.$$

Up to a permutation of the I_j 's, W is a symmetric function of the v_k 's. Therefore we permute the coordinates of \mathbf{v} with $\sigma \in \mathfrak{S}_M$ in such a way that $\mathbf{v}_\sigma \equiv (v_{\sigma(1)}, \dots, v_{\sigma(M)}) = t\mathbf{u}$, where $t > 0$ and $\mathbf{u} : \sum_{i=1}^M u_i^2 = 1$ has its coordinates ordered in the following way

$$\mathbf{u} = \underbrace{(u_1, \dots, u_1)}_{\lambda_1}, \dots, \underbrace{(u_s, \dots, u_s)}_{\lambda_s} \quad \text{with} \quad u_1 < u_2 < \dots < u_q \leq 0 < u_{q+1} < \dots < u_s. \quad (2.19)$$

Note that we have $\lambda_p > 0$ for $p = 1 \dots s$ and $M = \lambda_1 + \dots + \lambda_s$. The above permutation of the coordinates of \mathbf{v} results in a permutation of the integers $I_j \mapsto I_{\sigma(j)}$.

For further convenience we introduce the shorthand notations

$$w_k = \sum_{p=1}^k \lambda_p \quad \text{and} \quad J_\ell = \sum_{j=w_{\ell-1}+1}^{w_\ell} I_{\sigma(j)}, \quad (2.20)$$

and agree upon $w_0 = 0$. After some algebra, we recast W in the form

$$W(\mathbf{v}) = \sum_{k=1}^s \lambda_k \Phi(tu_k) + \frac{1}{2} \sum_{i,j=1}^s \lambda_i \lambda_j \Theta(tu_i - tu_j) + \pi(M-1)K \sum_{k=1}^s \lambda_k tu_k - 2\pi \sum_{k=1}^s tu_k J_k. \quad (2.21)$$

Here, we can already send $t \rightarrow +\infty$ and compute the asymptotic behavior of all functions without problem:

$$\Phi(tu_k) = \begin{cases} 2\pi\varphi_\infty tu_k + O(\ln t), & k \in \{1, \dots, q\}, \\ 2\pi(\varphi_\infty + N)tu_k + O(\ln t), & k \in \{q+1, \dots, s\}, \end{cases} \quad (2.22)$$

$$\sum_{i,j=1}^s \lambda_i \lambda_j \Theta(tu_i - tu_j) = 2\pi K t \sum_{1 \leq i < j \leq s} \lambda_i \lambda_j (u_j - u_i) + O(\ln t). \quad (2.23)$$

As a result, the asymptotics can be recast in the form

$$W(t\mathbf{u}) = 2\pi t \left(\sum_{k=1}^q u_k \tau_k^{(-)} + \sum_{k=q+1}^s u_k \tau_k^{(+)} \right) + O(\ln t), \quad (2.24)$$

where the two sequences $\tau_k^{(\pm)}$ read

$$\begin{aligned} \tau_k^{(-)} &= \lambda_k \left[\varphi_\infty + \frac{K}{2}(\lambda_k - 1) + K w_{k-1} \right] - J_k, \\ \tau_k^{(+)} &= \lambda_k \left[\varphi_\infty + N + MK - \frac{K}{2}(\lambda_k + 1) - K(M - w_k) \right] - J_k. \end{aligned}$$

The claim will follow as soon as we prove that the sum in the brackets in (2.24) is strictly positive. We first focus on the case where $u_q \neq 0$. Then the “(+)” and “(−)” partial sums are both strictly positive. Indeed we observe that $\sum_{k=1}^q u_k \tau_k^{(-)}$ can be re-cast as:

$$\sum_{k=1}^q u_k \tau_k^{(-)} = \sum_{k=1}^{q-1} (u_k - u_{k+1}) \left(\sum_{j=1}^k \tau_j^{(-)} \right) + u_q \sum_{j=1}^q \tau_j^{(-)}. \quad (2.25)$$

There, we have from the very definition of the ordering of the u_k 's that $u_k - u_{k+1} < 0$ and $u_q < 0$. Also

$$\begin{aligned} \sum_{j=1}^k \tau_j^{(-)} &= \varphi_\infty \sum_{p=1}^k \lambda_p + \frac{K}{2} \sum_{p=1}^k \lambda_p (\lambda_p - 1) + K \sum_{j < p}^k \lambda_p \lambda_j - \sum_{j=1}^k J_j \\ &= \varphi_\infty w_k - \frac{K}{2} w_k + \frac{K}{2} \sum_{j,p=1}^k \lambda_p \lambda_j - \sum_{j=1}^{w_k} I_{\sigma(j)} \\ &\leq \varphi_\infty w_k + \frac{K}{2} w_k (w_k - 1) - \frac{K}{2} w_k (w_k - 1) = \varphi_\infty w_k < 0 \end{aligned}$$

where we have used the *lhs* of (2.18) and the definition of w_k (2.20) so as to obtain the last line. It thus follows that $\sum_{k=1}^q u_k \tau_k^{(-)} > 0$.

It now remains to show that $\sum_{k=q+1}^s u_k \tau_k^{(+)} > 0$. Very similarly, we decompose

$$\sum_{k=q+1}^s u_k \tau_k^{(+)} = \sum_{k=q+2}^s (u_k - u_{k-1}) \left(\sum_{j=k}^s \tau_j^{(+)} \right) + u_{q+1} \cdot \sum_{j=q+1}^s \tau_j^{(+)}. \quad (2.26)$$

As $u_k - u_{k-1} > 0$ and $u_{q+1} > 0$, we ought to check the sign of the partial sums involving the $\tau_k^{(+)}$. We set $\tilde{w}_k = \sum_{j=k}^s \lambda_j = M - w_{k-1}$ and get

$$\sum_{j=k}^s \tau_j^{(+)} = \tilde{w}_k(\varphi_\infty + N + MK) - \frac{K}{2}\tilde{w}_k(\tilde{w}_k + 1) - \sum_{j=\tilde{w}_{k-1}+1}^M I_{\sigma(j)} \geq \tilde{w}_k\varphi_\infty + 1 > 0 . \quad (2.27)$$

The last inequality follows by applying the *rhs* part of (2.18).

Now we assume that $u_q = 0$. As \mathbf{u} lies on the unit sphere it has to have non-zero coordinates. This means that necessarily $s > 1$. In its turn this ensures that even if the contribution of u_q to the asymptotics is zero, there are still other u_ℓ 's such that taken as a whole, the sum in the brackets in (2.24) is strictly positive.

Therefore, provided that the M integers I_j , $j = 1, \dots, M$ satisfy the 2^M inequalities (2.18), we see that the potential $W(\mathbf{v})$ grows in all directions. This implies that it admits at least one local minimum, and hence that there exists a solution to the logarithmic Bethe equation (1.14) associated with these integers.

Reciprocally, suppose that one of the 2^M inequalities (2.18) is not satisfied. We distinguish between two cases. First suppose that there exists a subset \mathcal{J} of cardinality $r \in \{1, \dots, M\}$ such that the *lhs* part of inequality (2.18) is not satisfied. Then, as the inequalities only involve integers, we necessarily have

$$K \frac{r(r-1)}{2} - 1 \geq \sum_{k=1}^r I_{\sigma(k)} . \quad (2.28)$$

Here $\sigma \in \mathfrak{S}_M$ is a permutation that maps $\{1, \dots, r\}$ onto \mathcal{J} . We choose to send the vector \mathbf{v} to infinity along the ray

$$v_{\sigma(k)} = -\frac{t}{\sqrt{r}} , \text{ for } k = 1, \dots, r \quad \text{and} \quad v_{\sigma(k)} = 0 , \text{ for } k = r+1, \dots, M . \quad (2.29)$$

Then according to (2.24), we get

$$W(\mathbf{v}) = -\frac{2\pi t}{\sqrt{r}} \tau_1^{(-)} + O(\ln t) \quad (2.30)$$

where

$$\tau_1^{(-)} = r\varphi_\infty + \frac{K}{2}r(r-1) - \sum_{k=1}^r I_{\sigma(k)} \geq r\varphi_\infty + 1 > 0 . \quad (2.31)$$

This shows that W is unbounded from below and hence cannot have any minimum. Similarly, if there exists a subset \mathcal{J} of cardinality $r \in \{1, \dots, M\}$ such that the *rhs* of inequality (2.18) is not satisfied, then

$$\sum_{k=1}^r I_{\sigma(k)} \geq rMK - \frac{r(r+1)}{2}K + rN . \quad (2.32)$$

Here $\sigma \in \mathfrak{S}_M$ is any permutation that maps $\{1, \dots, r\}$ onto \mathcal{J} . We choose to send the vector \mathbf{v} to infinity along the ray

$$v_{\sigma(k)} = \frac{t}{\sqrt{r}} , \text{ for } k = 1, \dots, r \quad \text{and} \quad v_{\sigma(k)} = 0 , \text{ for } k = r+1, \dots, M . \quad (2.33)$$

Then according to (2.24), we get

$$W(\mathbf{v}) = \frac{2\pi t}{\sqrt{r}} \tau_2^{(+)} + O(\ln t) . \quad (2.34)$$

Paying attention to the conventions introduced in (2.19), we see that $\lambda_1 = M - r$, $\lambda_2 = r$ and thus $w_2 = M$,

$$\tau_2^{(+)} = r\varphi_\infty + r(N + MK) - \frac{K}{2}r(r+1) - \sum_{k=1}^r I_{\sigma(k)} \leq r\varphi_\infty < 0 . \quad (2.35)$$

This shows that W goes to $-\infty$ along that ray and the claim follows. \blacksquare

3 Number of independent solutions in the trigonometric case

3.1 Properties of factor modules

Factor modules over \mathbb{Z}^M provide a convenient language for describing set of integers that are identified modulo shifts as those appearing in theorem 3.

Let $\text{Mat}_M(\mathbb{Z})$ be the set of $M \times M$ matrices over the ring \mathbb{Z} . Given a matrix $G \in \text{Mat}_M(\mathbb{Z})$, $G_{kl} = \gamma_{kl}$, $\gamma_{kl} \in \mathbb{Z}$, one defines $\langle G \rangle$ to be the *submodule* in \mathbb{Z}^M generated by the rows of G . Then, the *factor-module* $\mathbb{G} \equiv \mathbb{Z}^M / \langle G \rangle$ is the set of equivalence classes of the integer vectors (I_1, \dots, I_M) differing by a vector from $\langle G \rangle$. In other words, \mathbb{G} is generated by the *generators* forming a column vector $\mathbf{e} = (e_1, \dots, e_m)^t$ subject to the *relations*

$$G\mathbf{e} = 0, \quad \text{or explicitly} \quad \sum_{n=1}^M \gamma_{mn} e_n = 0, \quad m = 1, \dots, M . \quad (3.1)$$

The factor module \mathbb{G} relevant for counting the solutions to the generalized Bethe equations, is given by a matrix G with entries

$$\gamma_{mn} = \begin{cases} a, & n = m, \\ b, & n \neq m, \end{cases} \quad \text{or} \quad \gamma_{mn} = a\delta_{mn} + b(1 - \delta_{mn}) , \quad (3.2)$$

provided that one takes $a = -K$ and $b = N + (M - 1)K$.

Indeed, the above factor-module \mathbb{G} is identical to the set of equivalence classes of the integer sequences \mathbf{I} with respect to the transformations (2.9).

The strategy we adopt for counting the number of solutions of the generalized Bethe equations (0.1) in the trigonometric case is as follows. We consider a factor module \mathbb{G} defined by the matrix G given in (3.2). We start by determining the total number of elements belonging to \mathbb{G} . Then, in order to relate $\#\mathbb{G}$ to the number of distinct solutions as described in theorem 3 we should discard all sequences $\mathbf{I} = \sum_{k=1}^M I_k e_k \in \mathbb{G}$ which have at least one repeating component $I_j = I_k$, for some $j \neq k$. Finally, as

the independent solutions of the generalized Bethe equations are in bijection with the monotonously growing sequences $I_1 < \dots < I_M$, we have to divide this last result by the number of allowed permutations so as to obtain the number of solutions.

Recall that the permutation group acts in a natural way on the module \mathbb{Z}^M :

$$\sigma : (I_1, \dots, I_M) \mapsto (I_{\sigma(1)}, \dots, I_{\sigma(M)}). \quad (3.3)$$

It is obvious that the submodule $\langle G \rangle$ given by the matrix G (3.2) is invariant under the above action of \mathfrak{S}_M . The induced action of \mathfrak{S}_M can be then defined on the factor-module \mathbb{G} .

An *orbit* of a point $\mathbf{I} = \sum_{j=1}^M I_j e_j \in \mathbb{G}$ is the set of all $\mathbf{I}_\sigma = \sum_{j=1}^M I_{\sigma(j)} e_j$ for $\sigma \in \mathfrak{S}_M$. A *regular*, or *generic* orbit is an orbit that has the maximal possible number of distinct points, that is $M!$. An example of points $\mathbf{I} = \sum_{k=1}^M I_k e_k$ not giving rise to regular orbits are the so-called diagonal points, that is points having at least two integers coordinates equal: $I_j = I_k$ for $j \neq k$. The regular orbits of the action of \mathfrak{S}_M on \mathbb{G} are described by the following lemma

Lemma 1. *The regular orbits of the factor module \mathbb{G} subordinate to the period matrix G (3.2) are generated by all elements $\mathbf{I} = \sum_{k=1}^M I_k e_k$ such that the integers I_j form monotonous sequences $I_1 < \dots < I_M$ and the vector (I_1, \dots, I_M) belongs to the fundamental domain \mathfrak{G} of \mathbb{G}*

$$\mathfrak{G} = \left\{ \mathbf{x} \in \mathbb{R}^M : x_i = t_i(a - b) + b \sum_{p=1}^M t_p, \quad t_i \in (0, 1) \right\}. \quad (3.4)$$

Proof. It is clear from the previous discussion that the regular orbits can only be given by off-diagonal elements. Moreover, one can always represent \mathbb{G} by points in its fundamental domain. Let $\mathbf{I} = \sum_{k=1}^M I_k e_k$ be defined in terms of a monotonous sequence of integers $I_1 < I_2 < \dots < I_M$ such that (I_1, \dots, I_M) lies in the fundamental domain of \mathbb{G} .

Assume that the \mathfrak{S}_M -orbit of \mathbf{I} in $\mathbb{G} = \mathbb{Z}^M / \langle G \rangle$ is not regular. Then, there exist at least two distinct permutations σ and σ' such that $\mathbf{I}_\sigma \equiv \mathbf{I}_{\sigma'}$, where $\mathbf{I}_\sigma = \sum_{p=1}^M I_{\sigma(p)} e_p$. In other words,

$$(I_{\sigma(1)}, \dots, I_{\sigma(M)}) = (I_{\sigma'(1)}, \dots, I_{\sigma'(M)}) + \sum_{j=1}^M \alpha_j (\gamma_{j1}, \dots, \gamma_{jM}), \quad (3.5)$$

for some $\alpha_j \in \mathbb{Z}$ and γ_{mn} given by (3.2). Applying the inverse permutation, we get

$$(I_1, \dots, I_M) = (I_{\sigma'\sigma^{-1}(1)}, \dots, I_{\sigma'\sigma^{-1}(M)}) + \sum_{j=1}^M \alpha_{\sigma^{-1}(j)} (\gamma_{\sigma^{-1}(j)\sigma^{-1}(1)}, \dots, \gamma_{\sigma^{-1}(j)\sigma^{-1}(M)}). \quad (3.6)$$

The period matrix G (3.2) can be recast as

$$G = (a - b)\text{id} + bx \cdot y^t,$$

where $x^t = y^t = (1, \dots, 1)$ are permutation-invariant vectors. Clearly, this implies that the period matrix is invariant under permutations, ie $\gamma_{\sigma(j)\sigma(k)} = \gamma_{jk}$. It thus follows that there exist integers $\beta_j \in \mathbb{Z}$ and a non-trivial permutation $\pi \in \mathfrak{S}_M$ such that

$$I_p = I_{\pi(p)} + (a - b)\beta_p + b \sum_{\ell=1}^M \beta_\ell. \quad (3.7)$$

Summing (3.7) over all the p 's we get that

$$(a + (M - 1)b) \sum_{\ell=1}^M \beta_\ell = 0 \quad \Rightarrow \quad \sum_{\ell=1}^M \beta_\ell = 0, \quad (3.8)$$

the latter stemming from the fact that the period matrix G is assumed to have a non-vanishing determinant, cf (3.17). Thus, we get an expression for β_p in terms of the parameters t_p describing the points I_p , namely

$$\beta_p = t_p - t_{\pi(p)} \in (-1, 1). \quad (3.9)$$

As $\beta_p \in \mathbb{Z}$, we necessarily have $\beta_p = 0$ for all p . As \mathbf{I} is an off-diagonal element $(I_1, \dots, I_M) = (I_{\pi(1)}, \dots, I_{\pi(M)})$ implies that $\pi = \text{id}$, which contradicts the fact that $\sigma \neq \sigma'$. \blacksquare

The above lemma leads to a complete description of the solutions of the generalized Bethe equations in the trigonometric case.

Proposition 8. *The solutions of the trigonometric generalized Bethe equations (0.1) are in a one-to-one correspondence with the regular orbits of the factor-module*

$$\mathbb{G} \equiv \mathbb{Z}^M / \langle G \rangle$$

with respect to the action of permutation group \mathfrak{S}_M .

In the remaining part of this Section, we will prove

Theorem 4. *For a matrix G of the form (3.2), where a and b are not necessarily given by (2.7), the number of regular orbits of the factor module $\mathbb{G} = \mathbb{Z}^M / \langle G \rangle$ is*

$$C_M^{[a,b]} = \frac{|a + (M - 1)b|}{M} \binom{|a - b| - 1}{M - 1}. \quad (3.10)$$

Taking $a = N + (M - 1)K$ and $b = -K$ we get that

Corollary 3. *The number of solutions to the generalized Bethe ansatz equation in the trigonometric case is given, for $N = 1$, by the Fuss-Catalan numbers $C_M^{(K,1)}$, and by*

$$C_M^{(K,N)} = \frac{N}{M} \binom{KM + N - 1}{M - 1} = \frac{N}{KM + N} \binom{KM + N}{M}, \quad (3.11)$$

for arbitrary N . Note that if $C(x) = \sum_{M=0}^{+\infty} C_M^{(K,1)} t^M$ is the generating function of the Fuss-Catalan numbers then $C^N(x) = \sum_{M=0}^{+\infty} C_M^{(K,N)} t^M$ is the generating function of the numbers $C_M^{(K,N)}$.

3.2 Number of elements in \mathbb{G}

Proposition 9. *Let \mathbb{G} be a factor module of \mathbb{Z}^M defined by some period matrix $G \in \text{Mat}_M(\mathbb{Z})$ such that $\det G \neq 0$, then \mathbb{G} is a finite set, and its cardinality, up to an eventual sign, equals to the determinant of G :*

$$\#\mathbb{G} = |\det G|. \quad (3.12)$$

Proof. It follows from elementary algebra that the matrix G admits a Smith normal form:

$$G = UDV, \quad U, D, V \in \text{Mat}_M(\mathbb{Z}), \quad D = \text{diag}(d_1, \dots, d_M). \quad (3.13)$$

and the matrices U and V are invertible (ie $\det U = \pm 1$, $\det V = \pm 1$).

Substituting (3.13) into the set of relations (3.1) determining \mathbb{G} we obtain

$$UDV\mathbf{e} = 0. \quad (3.14)$$

The matrix U being invertible, we multiply the relations by U^{-1} from the left and obtain an equivalent system of relations. On the other hand, due to the invertibility of V , the system of generators \mathbf{e} for the module \mathbb{Z}^M can be replaced by the equivalent system of generators $\mathbf{e}' = V\mathbf{e}$.

As a result, we get an equivalent description of the factor-module \mathbb{G} in terms of the generators \mathbf{e}' and the relations

$$D\mathbf{e}' = 0, \quad \text{ie} \quad d_m e'_m = 0, \quad \text{for} \quad m = 1, \dots, M. \quad (3.15)$$

Thus, the factor-module \mathbb{G} decomposes into a direct sum of rank 1 modules, each with the single generator e'_m and the single relation $d_m e'_m = 0$. Obviously, each component consists of $|d_m|$ elements that is equivalence classes whose representatives can be chosen, for example, as

$$0, e_m, 2e_m, 3e_m, \dots, (|d_m| - 1)e_m. \quad (3.16)$$

Thus the cardinality of \mathbb{G} equals $\prod_m |d_m| = |\det D|$. It remains to note that, by virtue of (3.13) $|\det G| = |\det D|$. \blacksquare

Corollary 4. *Let \mathbb{G} be the factor module defined by the period matrix G (3.2). Then*

$$\#\mathbb{G} = |\det G| = |(a + (M - 1)b)(a - b)^{M-1}|. \quad (3.17)$$

Proof. In the case of the matrix G of interest (3.2), $\det G$ can be computed thanks to the well-known formula for the determinant of rank-1 perturbation of a matrix. Let A be an arbitrary square matrix, x be a vector-column, and y^t be a vector-row. Then

$$\det(A + xy^t) = \det A + y^t \check{A} x, \quad (3.18)$$

where \check{A} is the adjoint matrix that is $A\check{A} = \det A$.

In the case of interest we have $G = A + xy^t$, where

$$A = (a - b) \text{id}_M, \quad x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad y^t = (b, \dots, b). \quad (3.19)$$

■

As we have already pointed out, formula (3.17), constitutes only the first step towards the answer; what we need to count are not all the points of \mathbb{G} but the regular orbits in respect to the action of the group \mathfrak{S}_M .

3.3 The diagonal submodules of \mathbb{G}

In order to describe the diagonal submodules of \mathbb{G} , we start by presenting an example.

Example. When $M = 2$, the period matrix

$$G = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad (3.20)$$

produces the system of relations

$$\begin{cases} ae_1 + be_2 = 0, \\ be_1 + ae_2 = 0. \end{cases} \quad (3.21)$$

Diagonals correspond to $I_1 = I_2 \equiv I_{12}$ that is to the points $I_{12}(e_1 + e_2)$. In other words, the element $e_1 + e_2$ generates a submodule, which we shall call a *diagonal* submodule. The relation for this submodule is obtained by adding together the equations (3.21):

$$(a + b)(e_1 + e_2) = 0. \quad (3.22)$$

The size of the submodule (the number of points on the diagonal) is obviously $|a + b|$.

Let \mathcal{P} be a partition of the set $\{1, \dots, M\}$ into J disjoint non-empty subsets $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$. To each partition \mathcal{P} we associate the diagonal submodule $\mathcal{D}_{\mathcal{P}}$ generated by J generators $e_{\mathcal{P}_j} \equiv \sum_{i \in \mathcal{P}_j} e_i$. Its matrix of relations is obtained from the matrix G by adding the rows labeled by $i \in \mathcal{P}_j$.

Example. For $M = 3$ there are 5 partitions of the set $\{1, 2, 3\}$: the 3-component one: $\{\{1\}, \{2\}, \{3\}\}$ corresponding to the original module $\langle G \rangle$, and 4 other partitions generating diagonal modules: three 2-component ones: $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$ and one 1-component: $\{\{1, 2, 3\}\}$.

The 2-component partition $\{\{1, 2\}, \{3\}\}$ gives rise to the generators $\{e_1 + e_2 \equiv e_{12}, e_3\}$ and relations $(a+b)e_{12} + 2be_3 = be_{12} + ae_3 = 0$, similarly for the rest two. The 1-component partition $\{\{1, 2, 3\}\}$ gives rise to a single generator $e_1 + e_2 + e_3 \equiv e_{123}$ and a single relation $(a + 2b)e_{123} = 0$.

Proposition 10. *Let $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$ be a partition of $\{1, \dots, M\}$. Then the associated diagonal submodule $D_{\mathcal{P}}$ is isomorphic to the diagonal submodule \mathcal{D}_{λ} corresponding to the partition $\mathcal{P}_1 = \{1, \dots, \lambda_1\}$, \dots , $\mathcal{P}_J = \{M - \lambda_J + 1, \dots, M\}$, where $\lambda_j = \#\mathcal{P}_j$ gives the number of elements in the partition \mathcal{P}_j .*

Proof. Taking the cardinalities $\#\mathcal{P}_j$ of the components of the partition \mathcal{P} of the set $\{1, \dots, M\}$ and rearranging them in the decreasing order we obtain a partition $\lambda =$

$\{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_J\}$ of the *number* M into J nonzero numbers: $|\lambda| \equiv \lambda_1 + \dots + \lambda_J = M$. Clearly, partitions \mathcal{P} corresponding to the same partition λ are related by permutations of the numbers $\{1, \dots, M\}$, and the corresponding diagonal submodules are isomorphic. ■

Corollary 5. *The number of elements in every diagonal submodule $\mathcal{D}_{\mathcal{P}}$ only depends on the number J of components of the partition $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$ and is given by*

$$\#\mathcal{D}_{\mathcal{P}} \equiv \Delta_J^M = (a + (M - 1)b)(a - b)^{J-1}. \quad (3.23)$$

Proof. As $\mathcal{D}_{\mathcal{P}}$ and \mathcal{D}_{λ} are isomorphic, it is enough to count the number of elements in \mathcal{D}_{λ} . The matrix elements of period matrix G_{λ} for the diagonal submodule \mathcal{D}_{λ} are given by

$$g_{jk}^{\lambda} = \delta_{jk}((a + (\lambda_j - 1)b) + (1 - \delta_{jk})\lambda_j b), \quad j, k = 1, \dots, J. \quad (3.24)$$

Rewriting (3.24) as

$$g_{jk}^{\lambda} = (a - b)\delta_{jk} + \lambda_j b, \quad (3.25)$$

we notice that G_{λ} decomposes into a sum of a diagonal matrix $A = (a - b)\text{id}_J$ and a rank 1 matrix xy^t , where $x_j = \lambda_j$ and $y_k = b$. Applying equation (3.18) we obtain the claim. ■

3.4 Elimination of diagonals and Stirling numbers: proof of theorem 4

We are now in position to prove Theorem 4.

The case $M = 2$ was analyzed at the beginning of the previous Section. There, one has the decomposition

$$2! C_2^{[a,b]} = |\det G_{11}| - |\det G_2| = |\Delta_2^2| - |\Delta_1^2| = |a + b|(|a - b| - 1). \quad (3.26)$$

For $M = 3$ we have 3 partitions of the set $\{1, 2, 3\}$ of the type $\lambda = (21)$. Subtracting the corresponding points count from the number of points in the original module \mathbb{G} we obtain

$$|\det G_{111}| - 3|\det G_{21}|.$$

Note, however, that each of the 3 diagonals $I_1 = I_2$, $I_1 = I_3$, $I_2 = I_3$ contains the common sub-diagonal $I_1 = I_2 = I_3$. We have thus subtracted it thrice. To compensate, we need to add it back twice. Finally, we get:

$$\begin{aligned} 3! C_3^{[a,b]} &= |\det G_{111}| - 3|\det G_{21}| + 2|\det G_3| \\ &= |\Delta_3^3| - 3|\Delta_2^3| + 2|\Delta_1^3| \\ &= |a + 2b|(|a - b| - 1)(|a - b| - 2). \end{aligned} \quad (3.27)$$

Proceeding similarly for $M > 3$ and using corollary 5 stating that $\det G_{\mathcal{P}}$ only depends on J we arrive to the following expansion:

$$\begin{aligned} M! C_M^{[a,b]} &= \sum_{J=1}^M (-1)^{M-J} s_{MJ} |\Delta_J^M| \\ &= |a + (M-1)b| \sum_{J=1}^M (-1)^{M-J} s_{MJ} |(a-b)^{J-1}|, \end{aligned} \quad (3.28)$$

where s_{MJ} are some combinatorial coefficients that *do not depend* on a and b .

To determine s_{MJ} we invoke their independence on a and b and set $b = 0$. Then, the factor-module \mathbb{G} becomes a hypercube having sides of length $|a|$. The regular orbits are labeled by the sequences $0 \leq I_1 < I_2 < \dots < I_M \leq |a| - 1$. The number of such points is obviously

$$C_M^{[a,0]} = \binom{|a|}{M} = \frac{|a|(|a|-1)\dots(|a|-M+1)}{M!}. \quad (3.29)$$

Setting $b = 0$ in (3.28) and substituting (3.29), we obtain its expansion into powers of $|a|$

$$|a|(|a|-1)\dots(|a|-M+1) = \sum_{J=1}^M (-1)^{M-J} s_{MJ} |a|^J. \quad (3.30)$$

The latter coincides with the definition of the (unsigned) *Stirling numbers of 1st kind* [11, 12]. In Knuth's notation [11]:

$$s_{MJ} = \left[\begin{matrix} M \\ J \end{matrix} \right]. \quad (3.31)$$

Returning to the general case (3.28) and using the definition of the Stirling numbers we obtain

$$\begin{aligned} M! C_M^{[a,b]} &= |a + (M-1)b| \sum_{J=1}^M (-1)^{M-J} \left[\begin{matrix} M \\ J \end{matrix} \right] |(a-b)^{J-1}| \\ &= \frac{|a + (M-1)b|}{|(a-b)|} |a-b| (|a-b|-1)\dots(|a-b|-M+1) \\ &= |a + (M-1)b| (|a-b|-1)\dots(|a-b|-M+1). \end{aligned}$$

The last expression coincides with (3.10). ■

4 Proof of theorem 2

In this Section we prove theorem 2. The proof is carried out by establishing a bijection between the solutions to the generalized Bethe equation in the trigonometric and rational cases.

One might expect that the solutions to the generalized Bethe equations in the trigonometric case with parameters $\{\varepsilon\eta_k\}$ and $\{\varepsilon\xi_k\}$, scale as $O(\varepsilon)$ in the $\varepsilon \rightarrow 0^+$ limit (up to

some possible shifts by $(\pi\mathbb{Z})^M$). If this is indeed the case, then the rational equations can be considered as a degeneration of the trigonometric ones. Of course, one should be rather careful in such a limiting procedure. When $\varepsilon \rightarrow 0^+$, it could be that some solutions to the trigonometric equations may not go to zero, and thus one might lose some solutions in the procedure. In particular, there might not be a one-to-one correspondence between the two sets of solutions. We prove that this does not happen: one can build, for ε small enough, a one-to-one correspondence between the solutions to the generalized Bethe equations in the trigonometric and rational cases.

From now on, we assume that the parameters $\{\xi_k\}$ and $\{\eta_k\}$ occurring in the trigonometric case are rescaled by $\varepsilon > 0$. Therefore, the trigonometric functions F and S are given by

$$F(u) = F_\infty \prod_{n=1}^N \frac{\sin(u + \varepsilon \xi_n)}{\sin(u + \varepsilon \bar{\xi}_n)} \quad \text{where } F_\infty = e^{-2i\pi\varphi_\infty}, \quad -1 < (M+1)\varphi_\infty < 0$$

$$S(u) = \prod_{n=1}^K \frac{\sin(u + \varepsilon \eta_n)}{\sin(u + \varepsilon \bar{\eta}_n)}.$$

In the following, the parameters $\{\eta_k\}_{k=1}^K$ and $\{\xi_r\}_{r=1}^N$ are to be considered fixed. The only varying parameter will be ε . We will call it the dilatation parameter. From now on, we shall *only focus* on the solutions \mathbf{v} to the trigonometric GBE belonging to the fundamental domain $(-\pi/2, \pi/2]$, ie for all $j \in \{1, \dots, M\}$, $v_j \in (-\pi/2, \pi/2]$. We also introduce the following definitions for the function φ_ε and θ_ε :

$$\varphi_\varepsilon(u) = 2\pi\varphi_\infty + \sum_{n=1}^N \chi(u + \varepsilon \operatorname{Re}(\xi_n), \varepsilon \operatorname{Im}(\xi_n)),$$

$$\theta_\varepsilon(u) = \sum_{j=1}^K \chi(u + \varepsilon \operatorname{Re}(\eta_j), \varepsilon \operatorname{Im}(\eta_j)),$$

where $\chi(u, c)$ is given by (2.1).

4.1 Rational constraints on the trigonometric integers

In this section we prove the following theorem

Theorem 5. *There exists ε_0 small enough such that any sequence I_1, \dots, I_M defined by a solution to the trigonometric GBAE (1.14) with dilatation parameter $\varepsilon \in (0, \varepsilon_0)$ satisfies the inequalities:*

$$Nr - 1 - K \frac{r(r+1)}{2} + KMr \geq \sum_{k \in \mathcal{J}} I_k \geq K \frac{r(r-1)}{2}, \quad \forall \mathcal{J} \subset \{1, \dots, M\}. \quad (4.1)$$

The main consequence of this theorem is that any solutions of the trigonometric GBE with sufficiently small inhomogeneities $\{\varepsilon \eta_k\}$, gives rise to a set of M integers (I_1, \dots, I_M)

satisfying the set of inequalities issued from the minimum condition of the rational potential. This means that there is at most as much solutions to the trigonometric GBE as there are solutions to the rational GBE.

To prove the theorem we need to establish two preparatory lemmas. We first observe that for ε small enough, any coordinates v_j of a solution \mathbf{v} to the trigonometric logarithmic Bethe equations stays uniformly away from the boundaries $\pm\pi/2$.

Lemma 2. *There exists an $\varepsilon_0 > 0$ and an $\alpha > 0$ such that given any solution $\mathbf{v} = (v_1, \dots, v_M)$ to the trigonometric logarithmic GBE (1.14) with dilatation parameter $\varepsilon \in (0, \varepsilon_0)$, one has*

$$v_j \in \left[-\frac{\pi}{2} + \alpha, \frac{\pi}{2} - \alpha\right] \quad \text{for } j = 1, \dots, M. \quad (4.2)$$

Proof. We prove the statement by contradiction. Assume that there exists a sequence $\varepsilon_n \rightarrow 0$ and a sequence $\mathbf{v}^{(n)}$ of solutions to the logarithmic GBE arising in the trigonometric case and corresponding to a dilatation parameter ε_n such that

$$v_1^{(n)} \xrightarrow{n \rightarrow +\infty} -\frac{\pi}{2} \quad \text{or} \quad v_M^{(n)} \xrightarrow{n \rightarrow +\infty} \frac{\pi}{2} \quad (4.3)$$

Here, we choose to work with solutions having ordered coordinates $-\pi/2 \leq v_1^{(n)} < \dots < v_M^{(n)} \leq \pi/2$. As the sequence $\mathbf{v}^{(n)}$ is bounded in norm, it admits a converging subsequence. We continue to denote this converging subsequence by $\mathbf{v}^{(n)}$. It fulfills

$$\begin{aligned} v_s^{(n)} &\xrightarrow{n \rightarrow +\infty} -\pi/2, & s \in \{1, \dots, \ell\} \\ v_s^{(n)} &\xrightarrow{n \rightarrow +\infty} v_s^{(\infty)} \in (-\pi/2, \pi/2), & s \in \{\ell+1, \dots, M-\ell'\} \\ v_s^{(n)} &\xrightarrow{n \rightarrow +\infty} \pi/2, & s \in \{M-\ell'+1, \dots, M\}. \end{aligned} \quad (4.4)$$

Due to (4.3), at least one of the $v_j^{(n)}$ has to converge to one of the boundaries points $\pm\pi/2$. Hence, $\ell + \ell' \geq 1$. Taking the product of the trigonometric GBE for $v_s^{(n)}$ where s runs through the set $\mathcal{S} = \{1, \ell\} \cup \{M-\ell'+1, \dots, M\}$ yields

$$1 = \prod_{s \in \mathcal{S}} F(v_s^{(n)}) \cdot \prod_{s \in \mathcal{S}} \prod_{\substack{k=1 \\ k \neq s}}^M S(v_s^{(n)} - v_k^{(n)}) = \prod_{s \in \mathcal{S}} F(v_s^{(n)}) \cdot \prod_{s \in \mathcal{S}} \prod_{\substack{k=1 \\ k \notin \mathcal{S}}}^M S(v_s^{(n)} - v_k^{(n)}). \quad (4.5)$$

In the last equality we made use of the space parity condition satisfied by S so as to cancel out the products $S(v_s^{(n)} - v_k^{(n)})S(v_k^{(n)} - v_s^{(n)})$ for $k, s \in \mathcal{S}$. It is easy to see that for $s \in \mathcal{S}$, $F(v_s^{(n)}) \rightarrow F_\infty$. As for $k \notin \mathcal{S}$, $v_k^{(n)}$ converges to some point in $(-\pi/2, \pi/2)$. We have,

$$v_s^{(n)} - v_k^{(n)} \xrightarrow{n \rightarrow +\infty} \delta v_{sk} \in (-\pi, 0) \cup (0, \pi) \quad \forall s \in \mathcal{S} \text{ and } k \in \{1, \dots, M\} \setminus \mathcal{S}. \quad (4.6)$$

In its turn, this means that $S(v_s^{(n)} - v_k^{(n)}) \rightarrow 1$. Thus, the $n \rightarrow +\infty$ limit of (4.5) yields

$$F_\infty^{\ell+\ell'} = e^{-2i\pi(\ell+\ell')\varphi_\infty} = 1. \quad (4.7)$$

As $1 \leq \ell + \ell' < M + 1$, this contradicts the fact that $-1 < (M + 1)\varphi_\infty < 0$. \blacksquare

We now observe that, if we take any two points v, w lying in $(-\pi/2 + \alpha, \pi/2 - \alpha)$ for some $\alpha > 0$, then θ_ε will be almost contained in the interval $[0, 2\pi]$, ie $\theta_\varepsilon(v - w) \in [-\nu, 2\pi + \nu]$, where $\nu \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Lemma 3. *Let $0 < \alpha < \pi/4$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$2\pi K - 2\pi \frac{\varphi_\infty}{M^2} > \theta_\varepsilon(v - w) > 2\pi \frac{\varphi_\infty}{M^2} \quad \forall v, w \in \left[-\frac{\pi}{2} + \alpha, \frac{\pi}{2} - \alpha\right]. \quad (4.8)$$

Here, we remind that $-1 < \varphi_\infty < 0$.

Proof. Let $\text{Im } \eta > 0$. Then, as χ is strictly increasing on the real axis, given $v, w \in [-\pi/2 + \alpha, \pi/2 - \alpha]$ we have

$$\chi(-\pi + 2\alpha + \varepsilon \text{Re } \eta, \varepsilon \text{Im } \eta) \leq \chi(v - w + \varepsilon \text{Re } \eta, \varepsilon \text{Im } \eta) \leq \chi(\pi - 2\alpha + \varepsilon \text{Re } \eta, \varepsilon \text{Im } \eta).$$

In the $\varepsilon \rightarrow 0^+$ limit, the *lhs* goes to zero whereas the *rhs* goes to 2π . Indeed, taking ε small enough so that $0 < \alpha - \varepsilon|\text{Re}(\eta)|$, one gets

$$\begin{aligned} \left| \chi(-\pi + 2\alpha + \varepsilon \text{Re}(\eta), \varepsilon \text{Im}(\eta)) \right| &= \left| \int_{-\frac{\pi}{2}}^{-\pi + 2\alpha + \varepsilon \text{Re}(\eta)} \frac{\sinh(2\varepsilon \text{Im}(\eta))}{\sin(u + i\varepsilon \text{Im}(\eta)) \sin(u - i\varepsilon \text{Im}(\eta))} du \right| \\ &\leq \frac{\sinh(2\varepsilon \text{Im}(\eta)) \cdot \pi/2}{\sin(\alpha + i\varepsilon \text{Im}(\eta)) \sin(\alpha - i\varepsilon \text{Im}(\eta))} \xrightarrow{\varepsilon \rightarrow 0^+} 0. \end{aligned}$$

Similarly, using the quasi-periodicity of χ we get

$$\begin{aligned} \left| \chi(\pi - 2\alpha + \varepsilon \text{Re}(\eta), \varepsilon \text{Im}(\eta)) - 2\pi \right| &= \left| \int_{-\frac{\pi}{2}}^{-2\alpha + \varepsilon \text{Re}(\eta)} \frac{\sinh(2\varepsilon \text{Im}(\eta))}{\sin(u + i\varepsilon \text{Im}(\eta)) \sin(u - i\varepsilon \text{Im}(\eta))} du \right| \xrightarrow{\varepsilon \rightarrow 0^+} 0. \end{aligned}$$

It thus follows that there exists an $\varepsilon_0 > 0$ such that for all

$$k \in \{1, \dots, K\} \text{ and } v, w \in \left[-\frac{\pi}{2} + \alpha, \frac{\pi}{2} - \alpha\right] \quad (4.9)$$

we have

$$\frac{2\pi\varphi_\infty}{KM^2} < \chi(v - w + \varepsilon \text{Re}(\eta_k), \varepsilon \text{Im}(\eta_k)) < 2\pi - \frac{2\pi\varphi_\infty}{KM^2}. \quad (4.10)$$

uniformly in $\varepsilon \in (0, \varepsilon_0)$. Summing these inequalities over k we get the claim. \blacksquare

We are now in position to prove the main result of this subsection:

Proof [*Theorem 5*]. We choose an $\varepsilon_0 > 0$ such that the conclusion of lemmas 2 and 3 hold simultaneously. Let ε be such that $\varepsilon_0 > \varepsilon > 0$ and \mathbf{v} be any solution of the trigonometric logarithmic GBE with parameters $\{\varepsilon\eta_k\}$ and $\{\varepsilon\xi_p\}$. Then, according to the conclusions

of lemma 2 there exists an α such that $v_j \in (-\pi/2 + \alpha, \pi/2 - \alpha)$. By virtue of lemma 3, we have

$$2\pi \frac{\varphi_\infty}{M^2} < \theta_\varepsilon(v_j - v_k) < 2\pi K - 2\pi \frac{\varphi_\infty}{M^2}. \quad (4.11)$$

Also, since $v_j \in (-\pi/2 + \alpha, \pi/2 - \alpha)$ and one can always assume ε to be small enough so that $\alpha - \varepsilon |\operatorname{Re}(\xi_k)| > 0$ for $k = 1, \dots, N$, we have

$$2\pi(N + \varphi_\infty) > \varphi_\varepsilon(v_j) > 2\pi\varphi_\infty. \quad (4.12)$$

Let $\mathcal{J} \subset \{1, \dots, M\}$ such that $\#\mathcal{J} = r$, then

$$\begin{aligned} 2\pi \sum_{k \in \mathcal{J}} I_k &= \sum_{k \in \mathcal{J}} \varphi_\varepsilon(v_k) + \sum_{k \in \mathcal{J}} \sum_{j \neq k} \theta_\varepsilon(v_k - v_j) \\ &= \sum_{\substack{j, k \in \mathcal{J} \\ j > k}} [\theta_\varepsilon(v_k - v_j) + \theta_\varepsilon(v_j - v_k)] + \sum_{k \in \mathcal{J}} \varphi_\varepsilon(v_k) + \sum_{k \in \mathcal{J}} \sum_{\substack{j=1 \\ j \notin \mathcal{J}}}^M \theta_\varepsilon(v_k - v_j) \\ &\geq 2\pi K \frac{r(r-1)}{2} + 2\pi r \varphi_\infty + 2\pi \frac{r(M-r)}{M^2} \varphi_\infty \\ &\geq 2\pi(r+1)\varphi_\infty + 2\pi K \frac{r(r-1)}{2}. \end{aligned}$$

It remains to use the bound $-1 < (r+1)\varphi_\infty < 0$ and invoke the fact that $\sum_{k \in \mathcal{S}} I_k$ is an integer so as to obtain the equality in the *rhs* of (4.1).

One gets the second set of inequalities in a similar way

$$\begin{aligned} 2\pi \sum_{k \in \mathcal{J}} I_k &= \sum_{\substack{j, k \in \mathcal{J} \\ j > k}} [\theta_\varepsilon(v_j - v_k) + \theta_\varepsilon(v_k - v_j)] + \sum_{k \in \mathcal{J}} \varphi(v_k) + \sum_{k \in \mathcal{J}} \sum_{j \notin \mathcal{J}} \theta_\varepsilon(v_k - v_j) \\ &< 2\pi K \frac{r(r-1)}{2} + r2\pi\varphi_\infty + 2\pi Nr + 2\pi K(M-r)r - 2\pi \frac{r(M-r)}{M^2} \varphi_\infty \\ &\leq 2\pi \left(Nr - 1 - K \frac{r(r+1)}{2} + KM r \right). \end{aligned}$$

When passing to the last line, we again have used the fact that $\sum_{k \in \mathcal{J}} I_k$ is an integer. ■

4.2 From the rational to the trigonometric solutions

One can visualize the generalized Bethe equation as the set of constraints defining the real valued zeroes \mathbf{v} with pairwise distinct coordinates of the mapping

$$\begin{aligned} \mathcal{Y}_{\{\eta\}, \{\xi\}} : \mathbb{C}^M &\rightarrow \mathbb{C}^M \\ \mathbf{z} &\mapsto (\mathcal{Q}_{\{\eta\}, \{\xi\}}(z_1 \mid \mathbf{z}), \dots, \mathcal{Q}_{\{\eta\}, \{\xi\}}(z_M \mid \mathbf{z})). \end{aligned}$$

We remind that the trigonometric Baxter polynomial $\mathcal{Q}_{\{\eta\}, \{\xi\}}(z_M \mid \mathbf{z})$ has been introduced in (1.7).

Proposition 11. *Let \mathbf{v} be a solution of the rational generalized Bethe equations with parameters $\{\eta_k\}$ and $\{\xi_k\}$. Then, there exists an $\varepsilon_{\mathbf{v}} > 0$ and a mapping $g_{\mathbf{v}}(\varepsilon)$ from the interval $(-\varepsilon_{\mathbf{v}}, \varepsilon_{\mathbf{v}})$ to some open neighborhood of 0 in \mathbb{R}^M such that $\varepsilon g_{\mathbf{v}}(\varepsilon)$ is a solution of the trigonometric generalized Bethe equations with parameters $\{\varepsilon\eta\}$ and $\{\varepsilon\xi\}$.*

Proof. The function \mathcal{G} given by

$$\begin{aligned} \mathcal{G} : (-1; 1) \times \mathbb{R}^M &\rightarrow \mathbb{C}^M \\ (\varepsilon, \mathbf{z}) &\mapsto i\varepsilon^{-N-MK} \mathcal{Y}_{\{\varepsilon\eta\}, \{\varepsilon\xi\}}(\varepsilon \mathbf{z}) \end{aligned}$$

is continuously differentiable and

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \mathbf{z} \rightarrow \mathbf{v}}} \frac{\partial}{\partial z_j} \cdot \mathcal{G}_k(\varepsilon, \mathbf{z}) = F_{\infty} \left\{ \prod_{r=1}^N (v_k + \xi_r) \cdot \prod_{m=1}^M \prod_{\ell=1}^K (v_k - v_m + \eta_{\ell}) \right\} \frac{\partial^2}{\partial z_j \partial z_k} \cdot W(\mathbf{z}) \Big|_{\mathbf{z}=\mathbf{v}} . \quad (4.13)$$

Here W is the strictly convex potential (1.18) appearing in the rational case. As any solution to the generalized Bethe equations is real, the product in the pre-factor in the *rhs* is non-zero. We have shown in proposition 4 that W has a positive defined Hessian

$$\det \left[\frac{\partial^2}{\partial z_j \partial z_k} \cdot W(\mathbf{z}) \right]_{\Big|_{\mathbf{z}=\mathbf{v}}} > 0, \quad (4.14)$$

therefore $\det_M [\partial_{z_k} \mathcal{G}_{\ell}(0, \mathbf{v})] \neq 0$ and we can apply the implicit function theorem to the mapping \mathcal{G} in the vicinity of the point $(0, \mathbf{v})$.

There exists open neighborhoods $(-\varepsilon_{\mathbf{v}}, \varepsilon_{\mathbf{v}})$ of 0, and $V_{\mathbf{v}}$ of \mathbf{v} and a \mathcal{C}^1 mapping $g_{\mathbf{v}} : (-\varepsilon_{\mathbf{v}}, \varepsilon_{\mathbf{v}}) \rightarrow V_{\mathbf{v}}$, such that $g_{\mathbf{v}}(0) = \mathbf{v}$ and

$$\{(\varepsilon, \mathbf{z}) \in (-\varepsilon_{\mathbf{v}}, \varepsilon_{\mathbf{v}}) \times V_{\mathbf{v}} : \mathcal{Y}_{\{\varepsilon\eta\}, \{\varepsilon\xi\}}(\varepsilon \mathbf{z}) = 0\} = \{(\varepsilon, g_{\mathbf{v}}(\varepsilon)) : \varepsilon \in (-\varepsilon_{\mathbf{v}}, \varepsilon_{\mathbf{v}})\} . \quad (4.15)$$

$g_{\mathbf{v}}(\varepsilon)$ is thus a solution of the trigonometric GBE with parameters $\{\varepsilon\eta\}$ and $\{\varepsilon\xi\}$. ■

Proposition 12. *Let \mathbf{v} and \mathbf{v}' be two distinct solutions of the rational GBE. Let $\varepsilon_0 = \min(\varepsilon_{\mathbf{v}}, \varepsilon_{\mathbf{v}'}) > 0$, where $\varepsilon_{\mathbf{v}}$ is as given in the above proposition. Then, for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $g_{\mathbf{v}}(\varepsilon) \neq g_{\mathbf{v}'}(\varepsilon)$.*

Proof. As $\varepsilon \mapsto g_{\mathbf{v}}(\varepsilon)$ is continuous, it follows that

$$\varphi_{\varepsilon}(\varepsilon[g_{\mathbf{v}}]_k(\varepsilon)) \quad \text{and} \quad \theta_{\varepsilon}(\varepsilon[g_{\mathbf{v}}]_k(\varepsilon) - \varepsilon[g_{\mathbf{v}'}]_k(\varepsilon)) \quad (4.16)$$

are all continuous in ε on $(-\varepsilon_0, \varepsilon_0)$. Hence, the combination

$$\varepsilon \mapsto \varphi_{\varepsilon}(\varepsilon[g_{\mathbf{v}}]_k(\varepsilon)) + \sum_{\substack{j=1 \\ j \neq k}}^M \theta_{\varepsilon}(\varepsilon[g_{\mathbf{v}}]_k(\varepsilon) - \varepsilon[g_{\mathbf{v}'}]_k(\varepsilon)) \quad (4.17)$$

is also continuous. However, for any ε small enough, $\varepsilon g_{\mathbf{v}}(\varepsilon)$ solves the trigonometric generalized Bethe equations. Hence, the function defined in (4.17) is integer valued. Being

a continuous function of ε , we deduce that it is constant. The value of this constant can, for instance, be determined by setting $\varepsilon = 0$. It thus follows that given two distinct solutions \mathbf{v} and \mathbf{v}' of the rational GBE, one obtains two solutions of the trigonometric GBE that are characterized by two distinct set of integers \mathbf{I} and \mathbf{I}' . Therefore these solutions cannot coincide. ■

Hence, for $0 < \varepsilon < \varepsilon_0$ the family $\{\varepsilon g_{\mathbf{v}}(\varepsilon)\}$, where \mathbf{v} solves the rational GBE, provides as many distinct solutions of the trigonometric GBE with parameters $\{\varepsilon\eta\}$ and $\{\varepsilon\xi\}$ as there are solutions to the rational GBE with parameters $\{\eta\}$ and $\{\xi\}$. We have just established that there are at least as many solutions to the trigonometric GBE as there are to the rational ones. This ends the proof of theorem 2.

5 Conclusion

In this article, we have studied a generalization of the Bethe equation where the S -matrix has several poles and zeroes in the complex plane. We have characterized the set of solutions in the so-called repulsive regime. In particular, we have provided a thorough count of the number of solutions to these equations. We showed that this number is given or closely related to the Fuss-Catalan numbers. This allowed us to establish two yet-unknown combinatorial interpretations of these numbers. On the one hand, we have shown that the Fuss-Catalan numbers count the number of integers satisfying certain inequalities and on the other hand they give the number of regular orbits in certain factor modules.

For the moment, such generalized Bethe equation do not correspond to any integrable model. We plan to investigate integrable models giving rise to such generalized Bethe equations in the future. It also seems extremely appealing to analyse the combinatorial identities arising from other classes of Bethe equations then those given in (0.1). For instance, estimating the number of solutions to the multi-pole generalizations of Gaudin-type equations should also lead to combinatorial problems for counting integers subject to sets of constraints.

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References

- [1] H. Bethe, “On the theory of metals: Eigenvalues and Eigenfunctions of a linear chain of atoms.”, *Zeitschrift für Physik* **71** (1931), 205–226.

- [2] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, 1984.
- [3] N. M. Bogoliubov, A. G. Izergin, and V. E. Korepin, “*Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz.*”, Cambridge monographs on mathematical physics, 1993.
- [4] L.D. Faddeev, “*How Algebraic Bethe Ansatz works for integrable model.*”, Les Houches lectures (1996).
- [5] N. Reshetikhin, *A method of functional equations in the theory of exactly solvable quantum systems*, Lett. Math. Phys. **7** (1983), no. 3, 205–213.
- [6] V. O. Tarasov, *Algebraic Bethe Ansatz for the Izergin-Korepin R matrix*, Theor. Math. Phys. **76** (1988), no. 2, 793–803.
- [7] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, J. High Energy Phys. **05** (2004), 024.
- [8] T. Bargheer, N. Beisert, and F. Loebbert, *Long-range deformations for integrable spin chains*, J. Phys. A: Math. Gen. **42** (2009), 285205.
- [9] L. D. Faddeev and L. A. Takhtadzhan, “*Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model.*”, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. **109** (1981), 134–178.
- [10] A. N. Kirillov and Reshetikhin N. Yu., *The Yangians, Bethe Ansatz and combinatorics*, Lett. Math. Phys. **12** (1986), no. 3, 199–208.
- [11] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Reading, Massachusetts: Addison-Wesley, 1994.
- [12] R. P. Stanley, “*Enumerative Combinatorics.*”, Cambridge University Press, vol. 1, Cambridge Studies in Advanced Mathematics, 1997.
- [13] P. Hilton and J. Pedersen, “*Catalan numbers, their generalizations and their uses.*”, Math. Intelligencer **13** (1991), 64–75.
- [14] R. P. Stanley, “*Catalan Addendum*”, web: <http://www-math.mit.edu/~rstan/ec/catadd.pdf>.